Example 2 (Bi-double cover)

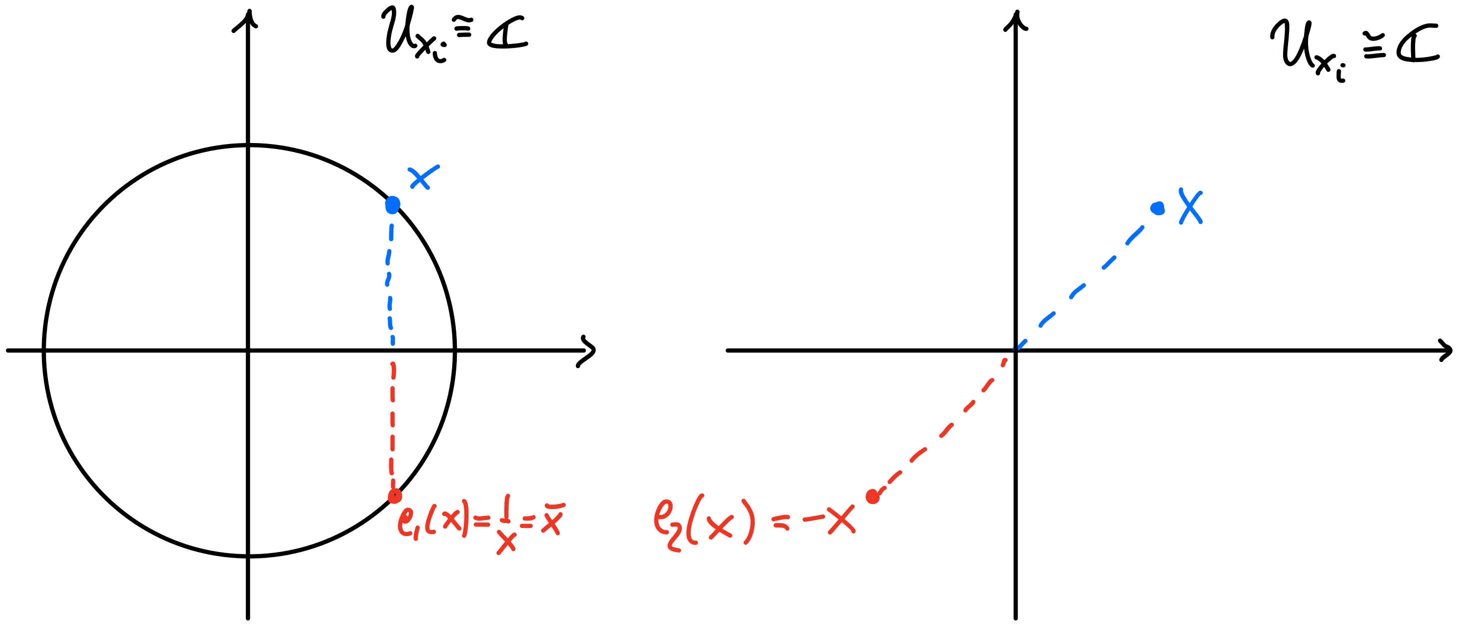
We take $X = \mathbb{P}^1(x_0, x_1)$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e_1, e_2 \rangle$ (two gens. of G)

$\bar{0} := Id_X$, $\quad e_1 : X \longrightarrow X, \; [x_0, x_1] \mapsto [x_1, x_0]$, $\quad e_2 : X \longrightarrow X, \; [x_0, x_1] \mapsto [x_0, -x_1]$

On $U_{x_i} = \{x_i \neq 0\} \subseteq X$ these two maps are respectively the inverse and opposite maps:

$e_1 : x \longmapsto \frac{1}{x} \; (= \bar{x} \text{ on } S^1)$ $\qquad$ where $x := \frac{x_j}{x_i}$.

$e_2 : x \longmapsto -x$



From the pictures one see that the points with no trivial stabilizer are six:

$[1, 0], [0, 1] \qquad Stab = \langle e_2 \rangle$

$[1, 1], [1, -1] \qquad Stab = \langle e_1 \rangle$

$[1, i], [1, -i] \qquad Stab = \langle e_1 + e_2 \rangle$

The action of G on X defines the bi-double quotient

$$\pi : X \longrightarrow \mathcal{Y} := \mathbb{P}^1(z_0, z_1)$$
$$[x_0, x_1] \longmapsto [x_0^4 + x_1^4, x_0^2 x_1^2]$$

Locally, $\pi : \pi^{-1}(U_{z_1}) \to U_{z_1}$, instead

$$x \longmapsto x^2 + \frac{1}{x^2}$$

locally $\pi : \pi^{-1}(U_{z_0}) \to U_{z_0}$ is $x \mapsto -\dfrac{x^2(x-1)^2}{x^4-(x-1)^4}$

where $x := \dfrac{x_0}{x_0 - a x_1}$, a is the first four root of -1.

Indeed a local chart of $\pi^{-1}(U_{z_0})$ is

$$\pi^{-1}(U_{z_0}) \xrightarrow{\sim} \mathbb{C}$$
$$[x_0, x_1] \longmapsto \frac{x_0}{x_0 - a x_1}$$

with inverse computed as follows:

$$x := \frac{x_0}{x_0 - a x_1} \Rightarrow x = \frac{x_0 - a x_1 + a x_1}{x_0 - a x_1} = 1 + a \frac{x_1}{x_0 - a x_1}$$

$$\Rightarrow \frac{1}{a}(x-1) = \frac{x_1}{x_0 - a x_1}$$

$$\Rightarrow \quad \mathbb{C} \xrightarrow{\sim} \pi^{-1}(U_{z_0})$$
$$x \longmapsto \left(x, \tfrac{1}{a}(x-1)\right)$$

Thus, locally π is

$$\mathbb{C} \xrightarrow{\sim} \pi^{-1}(U_{z_0}) \xrightarrow{\pi} U_{z_0} \xrightarrow{\quad\sim\quad} \mathbb{C}$$
$$x \longmapsto \left(x, \tfrac{1}{a}(x-1)\right) \longmapsto \left(x^4-(x-1)^4, \tfrac{1}{a^2}x^2(x-1)^2\right) \longmapsto \frac{1}{a^2}\frac{x^2(x-1)^2}{x^4-(x-1)^4}$$

Let us find the ramification locus of π:
Locally around $\pi^{-1}(\mathcal{U}_{z_1})$ we have

$$d\pi_x = \frac{d}{dx}\left(x^2+\frac{1}{x^2}\right) = 2x - 2\cdot\frac{1}{x^3} = 2\,\frac{x^4-1}{x^3} = 0 \iff$$

$$x^4 = 1 \iff x = 1, i, -1, -i$$
$$\iff [1,1], [i,1], [-i,1], [1,-1]$$

Let us see if there are others ram. points in the other chart $\pi^{-1}(\mathcal{U}_{z_0})$:

$$d\pi_x = \frac{d}{dx}\left(\frac{1}{a^2}\,\frac{x^2(x-1)^2}{x^4-(x-1)^4}\right) = \frac{1}{a^2}\,\frac{2x(2x^5-6x^4+10x^3-10x^2+5x-1)}{(4x^3-6x^2+4x-1)^2}$$

$$= 0$$

$$\iff x = 0, 1, \frac{1}{1-a}, \frac{a}{a-1}, \frac{1}{1-a^3}, \frac{a^3}{a^3-1}$$

$$\iff \begin{array}{l} [0,1], [1,0] \text{ new points} \\ \left.\begin{array}{l}[i,1], [i,-1] \\ [1,1], [1,-1]\end{array}\right\} \text{already obtained using the other chart.} \end{array}$$

$$\Rightarrow R = \mathrm{Ram}(\pi) = [1,0] + [0,1] + [i,1] + [i,-1] + [1,1] + [1,-1]$$
$$= R_{e_2} + R_{e_1+e_2} + R_{e_1}$$

$$\Rightarrow \underset{\substack{\Downarrow \\ \text{Reduced} \\ \text{branch locus}}}{D} = \pi(R) = \underset{\substack{\| \\ D_{e_2}}}{[1,0]} + \underset{\substack{\| \\ D_{e_1+e_2}}}{[2,-1]} + \underset{\substack{\| \\ D_{e_1}}}{[2,1]}$$

Let us consider now the sheaf $\pi_* \mathcal{O}_X$ on $\mathcal{Y}$, we want to prove that it is a locally free sheaf of rank 4 on $\mathcal{Y}$.

We choose the coordinate charts $\mathcal{U}_{z_0}$ and $\mathcal{U}_{z_1}$ on $\mathcal{Y}$:

$$\pi_* \mathcal{O}_X(\mathcal{U}_{z_1}) = \mathcal{O}_X(\pi^{-1}(\mathcal{U}_{z_1})) = \mathcal{O}_X(\mathcal{U}_{x_0} \cap \mathcal{U}_{x_1}) = \mathbb{C}[x, \tfrac{1}{x}]$$

where $x := \dfrac{x_1}{x_0}$.

By construction, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts naturally on $\pi_* \mathcal{O}_X(\mathcal{U}_{z_1})$ sending
$$e_1 : x \longmapsto \tfrac{1}{x}$$
$$e_2 : x \longmapsto -x$$

Thus, we have a representation of G on the space $\mathbb{C}[x, \frac{1}{x}]$. Let us determine its isotypic components W^η, $\eta \in \mathrm{Irr}(G)$ using Reynold Operator. Given $p(x) \in \mathbb{C}[x, \frac{1}{x}]$, then

$$\pi_0(p) = \tfrac{1}{4}\left(p(x) + p(\tfrac{1}{x}) + p(-x) + p(-\tfrac{1}{x})\right) \in \underset{\substack{\parallel \\ W^0 = \mathcal{O}_{\mathcal{Y}}(\mathcal{U}_{z_1})}}{\mathbb{C}[x^2 + \tfrac{1}{x^2}]}$$

$$\pi_{\varepsilon_1}(p) = \tfrac{1}{4}\left(p(x) - p(\tfrac{1}{x}) + p(-x) - p(-\tfrac{1}{x})\right) \in \mathbb{C}[x^2 + \tfrac{1}{x^2}] \cdot (x^2 - \tfrac{1}{x^2})$$

$$\pi_{\varepsilon_2}(p) = \frac{1}{4}\left(p(x) + p\left(\tfrac{1}{x}\right) - p(-x) - p\left(-\tfrac{1}{x}\right)\right) \in \mathbb{C}\left[x^2+\tfrac{1}{x^2}\right]\cdot\left(x+\tfrac{1}{x}\right)$$

$$\pi_{\varepsilon_1+\varepsilon_2}(p) = \frac{1}{4}\left(p(x) - p\left(\tfrac{1}{x}\right) - p(-x) + p\left(-\tfrac{1}{x}\right)\right) \in \mathbb{C}\left[x^2+\tfrac{1}{x^2}\right]\cdot\left(x-\tfrac{1}{x}\right)$$

Clearly, $p(x) = \pi_0(p) + \pi_{\varepsilon_1}(p) + \pi_{\varepsilon_2}(p) + \pi_{\varepsilon_1+\varepsilon_2}(p)$, so we obtain the decomposition

$$\begin{aligned}\pi_*\mathcal{O}_X(U_{z_1}) &= \mathbb{C}\left[x^2+\tfrac{1}{x^2}\right]\cdot 1 \oplus \mathbb{C}\left[x^2+\tfrac{1}{x^2}\right]\cdot\left(x^2-\tfrac{1}{x^2}\right) \oplus \mathbb{C}\left[x^2+\tfrac{1}{x^2}\right]\cdot\left(x+\tfrac{1}{x}\right)\\ &\quad\oplus \mathbb{C}\left[x^2+\tfrac{1}{x^2}\right]\cdot\left(x-\tfrac{1}{x}\right)\\ &\cong \mathcal{O}_Y(U_{z_1})\cdot \underset{f_0}{\overset{\text{inv. fct of char } 0}{1}} \oplus \mathcal{O}_Y(U_{z_1})\underset{f_{\varepsilon_1}}{\overset{\text{inv. fct of char } \varepsilon_1}{\left(x^2-\tfrac{1}{x^2}\right)}} \oplus \mathcal{O}_Y(U_{z_1})\underset{f_{\varepsilon_2}}{\overset{\text{inv. fct of char } \varepsilon_2}{\left(x+\tfrac{1}{x}\right)}}\\ &\quad\oplus \mathcal{O}_Y(U_{z_1})\cdot \underset{\text{inv. fct of char } \varepsilon_1+\varepsilon_2,\ \text{called } f_{\varepsilon_1+\varepsilon_2}}{\left(x-\tfrac{1}{x}\right)}\end{aligned}$$

What does it happen in the other chart?

$$\pi_*\mathcal{O}_X(U_{z_0}) = \mathcal{O}_Y(\{x_0^4+x_1^4\neq 0\}) = \mathbb{C}\left[t,\tfrac{1}{t}, w, \tfrac{1}{w}\right],$$

where $t := \dfrac{x_0 - a x_1}{x_0 + a x_1}$ and $w := -\dfrac{x_0 + a^3 x_1}{x_0 - a^3 x_1} = a^2\cdot\dfrac{t+a^2}{t-a^2}$

The action of G induces the following action on $\pi_*\mathcal{O}_X(U_{z_0})$:

$$e_1 : t \mapsto w, \qquad e_2 : t \mapsto \frac{1}{t}$$

Let us determine the isotypic components W^η, $\eta\in \mathrm{Irr}(G)$:

$$\pi_0(t) = \frac{1}{4}\left(t + w + \frac{1}{t} + \frac{1}{w}\right), \quad \pi_{\varepsilon_2}(t) = \frac{1}{4}\left(t + w - \frac{1}{t} - \frac{1}{w}\right)$$

$$\pi_{\varepsilon_1}(t) = \frac{1}{4}\left(t - w + \frac{1}{t} - \frac{1}{w}\right), \quad \pi_{\varepsilon_1+\varepsilon_2}(t) = \frac{1}{4}\left(t - w - \frac{1}{t} + \frac{1}{w}\right)$$

This suggests the following decomposition:

$$\mathbb{C}[t, w, \tfrac{1}{t}, \tfrac{1}{w}] = \mathbb{C}[t+w+\tfrac{1}{t}+\tfrac{1}{w}]\cdot 1 \oplus \mathbb{C}[t+w+\tfrac{1}{t}+\tfrac{1}{w}]\cdot \overbrace{\left(t - w + \tfrac{1}{t} - \tfrac{1}{w}\right)}^{f_{\varepsilon_1}}$$

$$\mathbb{C}[t+w+\tfrac{1}{t}+\tfrac{1}{w}]\cdot \underbrace{\left(t + w - \tfrac{1}{t} - \tfrac{1}{w}\right)}_{f_{\varepsilon_2}} \oplus \mathbb{C}[t+w+\tfrac{1}{t}+\tfrac{1}{w}]\cdot \underbrace{\left(t - w - \tfrac{1}{t} + \tfrac{1}{w}\right)}_{f_{\varepsilon_1+\varepsilon_2}}$$

and clearly $t + w + \frac{1}{t} + \frac{1}{w} = 8a^2\frac{x_0^2 x_1^2}{x_0^4 + x_1^4}$, so

$$\mathbb{C}[t+w+\tfrac{1}{t}+\tfrac{1}{w}] \cong \mathcal{O}_Y(U_{z_0})$$

We have seen that $\pi_* \mathcal{O}_X$ is a locally-free sheaf of rank 4. Let us find the cocycles of the associated rank 4 vector bundle:

$$\bigoplus_{i=1}^{4} \mathcal{O}_Y(U_{z_0} \cap U_{z_1}) \xrightarrow{\phi_0^{-1}} \pi_* \mathcal{O}_X(U_{z_0} \cap U_{z_1}) \xrightarrow{\quad \phi_1 \quad} \bigoplus_{i=1}^{4} \mathcal{O}_Y(U_{z_0} \cap U_{z_1})$$

$$(a_1, a_2, a_3, a_4) \longmapsto a_1 \cdot 1 + a_2 \cdot f_{\varepsilon_1} + a_3 f_{\varepsilon_2} + a_4 f_{\varepsilon_1+\varepsilon_2} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{f_{\varepsilon_1}}{g_{\varepsilon_1}} & 0 & 0 \\ 0 & 0 & \frac{f_{\varepsilon_2}}{g_{\varepsilon_2}} & 0 \\ 0 & 0 & 0 & \frac{f_{\varepsilon_1+\varepsilon_2}}{g_{\varepsilon_1+\varepsilon_2}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$\| $$

$$a_1 \cdot 1 + a_2 \cdot \frac{f_{\varepsilon_1}}{g_{\varepsilon_1}} \cdot g_{\varepsilon_1} + a_3 \frac{f_{\varepsilon_2}}{g_{\varepsilon_2}} + a_4 \frac{f_{\varepsilon_1+\varepsilon_2}}{g_{\varepsilon_1+\varepsilon_2}}$$

they are multiples of $\frac{x_0^4 + x_1^4}{x_0^2 x_1^2} = \frac{z_0}{z_1}$

$$\Rightarrow g_{01} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z_0}{z_1} & 0 & 0 \\ 0 & 0 & \frac{z_0}{z_1} & 0 \\ 0 & 0 & 0 & \frac{z_0}{z_1} \end{pmatrix}$$

This proves

$$\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{i=1}^{3} \mathcal{O}_Y(-1) =: \mathcal{L}_0 \oplus \mathcal{L}_{\varepsilon_1}^{-1} \oplus \mathcal{L}_{\varepsilon_2}^{-1} \oplus \mathcal{L}_{\varepsilon_1+\varepsilon_2}^{-1}$$

$\mathcal{O}_Y$-submodules of $\pi_*\mathcal{O}_X$ corresponding to the invariant functions of character ε_1, ε_2, $\varepsilon_1+\varepsilon_2$ respectively.

We have the following sections of the pullback $\mathcal{L}_{\varepsilon_1}, \mathcal{L}_{\varepsilon_2}, \mathcal{L}_{\varepsilon_1+\varepsilon_2}$ on X:

$$s_{\varepsilon_1} = \{ (\pi^{-1}(U_{z_0}), g_{\varepsilon_1}), \; (\pi^{-1}(U_{z_1}), f_{\varepsilon_1}) \}$$

$$s_{\varepsilon_2} = \{ (\pi^{-1}(U_{z_0}), g_{\varepsilon_2}), \; (\pi^{-1}(U_{z_1}), f_{\varepsilon_2}) \}$$

$$s_{\varepsilon_1+\varepsilon_2} = \{ (\pi^{-1}(U_{z_0}), g_{\varepsilon_1+\varepsilon_2}), \; (\pi^{-1}(U_{z_1}), f_{\varepsilon_1+\varepsilon_2}) \}$$

A global section of $\mathcal{L}_{\varepsilon_1} \otimes \mathcal{L}_{\varepsilon_2} \otimes \mathcal{L}_{\varepsilon_1+\varepsilon_2}^{-1}$ is then

$$\frac{s_{\varepsilon_1} \cdot s_{\varepsilon_2}}{s_{\varepsilon_1+\varepsilon_2}} = \left\{ \left(\pi^{-1}(U_{z_0}), \frac{g_{\varepsilon_1} \cdot g_{\varepsilon_2}}{g_{\varepsilon_1+\varepsilon_2}}\right), \left(\pi^{-1}(U_{z_1}), \frac{f_{\varepsilon_1} \cdot f_{\varepsilon_2}}{f_{\varepsilon_1+\varepsilon_2}}\right) \right\}$$

$$= \left\{ \left(\pi^{-1}(U_{z_0}), 1 + 2\frac{z_1}{z_0}\right), \left(\pi^{-1}(U_{z_1}), \frac{z_0}{z_1} + 2\right) \right\}$$

Thus the divisor associated to this section is

$$z\left(1+2\frac{z_1}{z_0}\right) = [2,-1] = D_{e_1+e_2}$$
$$= z\left(\frac{z_0}{z_1}+2\right)$$

We have proved that

$$L_{\varepsilon_1} + L_{\varepsilon_2} - L_{\varepsilon_1+\varepsilon_2} \equiv D_{e_1+e_2}.$$

In a similar way we can deduce Pardini Equations:

$$2L_{\varepsilon_1} \equiv D_{e_1} + D_{e_1+e_2}, \quad 2L_{\varepsilon_2} \equiv D_{e_2} + D_{e_1+e_2}, \quad 2L_{\varepsilon_1+\varepsilon_2} \equiv D_{e_1} + D_{e_2}$$

$$L_{\varepsilon_1} + L_{\varepsilon_2} \equiv L_{\varepsilon_1+\varepsilon_2} + D_{e_1+e_2}, \qquad L_{\varepsilon_1} + L_{\varepsilon_1+\varepsilon_2} \equiv L_{\varepsilon_2} + D_{e_1},$$
$$L_{\varepsilon_2} + L_{\varepsilon_1+\varepsilon_2} \equiv L_{\varepsilon_1} + D_{e_2}.$$

Let us consider the vector bundle

$$\pi' : V(L_{\varepsilon_1} \oplus L_{\varepsilon_2} \oplus L_{\varepsilon_1+\varepsilon_2}) \to Y$$

with local coordinates:

$(z := \frac{z_j}{z_i}, y^i_{\varepsilon_1}, y^i_{\varepsilon_2}, y^i_{\varepsilon_1+\varepsilon_2})$ on $(\pi')^{-1}(\mathcal{U}_{z_i})$. There is a natural action of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on $V(\ )$:

$$(z, y^i_{\varepsilon_1}, y^i_{\varepsilon_2}, y^i_{\varepsilon_1+\varepsilon_2}) \xmapsto{g} (z, \varepsilon_1(g)\, y^i_{\varepsilon_1}, \varepsilon_2(g)\, y^i_{\varepsilon_2}, (\varepsilon_1+\varepsilon_2)(g)\, y^i_{\varepsilon_1+\varepsilon_2}), \ g \in G.$$

Pardini equations suggest to consider the curve

$$X' \cap (\pi')^{-1}(U_{z_i}) := \left\{ (z, y^i_{\varepsilon_1}, y^i_{\varepsilon_2}, y^i_{\varepsilon_1+\varepsilon_2}) \;\middle|\; \begin{array}{l} (y^i_{\varepsilon_1})^2 = f^i_{e_1} f^i_{e_1+e_2},\ (y^i_{\varepsilon_2})^2 = f^i_{e_2} f^i_{e_1+e_2} \\ (y^i_{\varepsilon_1+\varepsilon_2})^2 = f^i_{e_1} f^i_{e_2},\ y^i_{\varepsilon_1} y^i_{\varepsilon_2} = y^i_{\varepsilon_1+\varepsilon_2} f^i_{e_1+e_2} \\ y^i_{\varepsilon_1} y^i_{\varepsilon_1+\varepsilon_2} = y^i_{\varepsilon_2} f^i_{e_1} \\ y^i_{\varepsilon_2} y^i_{\varepsilon_1+\varepsilon_2} = y^i_{\varepsilon_1} f^i_{e_2} \end{array} \right\}$$

$$= \left\{ (z, y^i_{\varepsilon_1}, y^i_{\varepsilon_2}, y^i_{\varepsilon_1+\varepsilon_2}) \;\middle|\; \mathrm{rk} \begin{pmatrix} f^i_{e_1} & y^i_{\varepsilon_1} & y^i_{\varepsilon_1+\varepsilon_2} \\ y^i_{\varepsilon_1} & f^i_{e_1+e_2} & y^i_{\varepsilon_2} \\ y^i_{\varepsilon_1+\varepsilon_2} & y^i_{\varepsilon_2} & f^i_{e_2} \end{pmatrix} \leq 1 \right\}$$

where $f^i_{e_1}, f^i_{e_2}, f^i_{e_1+e_2}$ are the polynomials on Z whose zero locus are the points D_{e_1}, D_{e_2} and $D_{e_1+e_2}$

(for instance, $f^0_{e_1} = (1 - 2\frac{z_1}{z_0})$, $f^1_{e_1} = (\frac{z_0}{z_1} - 2)$ ecc..)

Thus, $\pi' : X' \to Y$ is a bidouble cover with Galois group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, that only depends on the data $\{D_{e_1}, D_{e_2}, D_{e_1+e_2}\}$ and $\{L_{\varepsilon_1}, L_{\varepsilon_2}, L_{\varepsilon_1+\varepsilon_2}\}$ of Y.

Finally, X and X' are isomorphic:

$$\Psi : X \to X'$$
$$p \mapsto (\pi(p), s_{\varepsilon_1}(p), s_{\varepsilon_2}(p), s_{\varepsilon_1+\varepsilon_2}(p))$$

Furthermore, in our specific case we have

$$V(\mathcal{L}_{\varepsilon_1} \oplus \mathcal{L}_{\varepsilon_2} \oplus \mathcal{L}_{\varepsilon_1+\varepsilon_2}) = \mathbb{P}^4(z_0, z_1, y_{\varepsilon_1}, y_{\varepsilon_2}, y_{\varepsilon_1+\varepsilon_2}) \setminus \{z_0 = z_1 = 0\}$$

$$X' = \left\{ (z_0, z_1, y_{\varepsilon_1}, y_{\varepsilon_2}, y_{\varepsilon_1+\varepsilon_2}) \in \mathbb{P}^4 \;\middle|\; \mathrm{rk}\begin{pmatrix} f_{e_1} & y_{\varepsilon_1} & y_{\varepsilon_1+\varepsilon_2} \\ y_{\varepsilon_1} & f_{e_1+e_2} & y_{\varepsilon_2} \\ y_{\varepsilon_1+\varepsilon_2} & y_{\varepsilon_2} & f_{e_2} \end{pmatrix} \leq 1 \right\}$$

and the isomorphism Ψ is

$$[x_0, x_1] \longmapsto [x_0^4 + x_1^4, x_0^2 x_1^2, (x_0^4 - x_1^4), (x_0^2 + x_1^2) x_0 x_1, (x_0^2 - x_1^2) x_0 x_1]$$

where $f_{e_1} = z_0 - 2z_1$, $f_{e_2} = z_1$, $f_{e_1+e_2} = z_0 + 2z_1$.

Example 3 (S_3- cover)

$$\langle \tau, \sigma \mid \tau^2 = \sigma^3,\ \tau\sigma = \sigma^2\tau \rangle$$

Let us consider $G = S_3 = \langle \tau, \sigma \rangle$ and the action on $X = \mathbb{P}^1$:

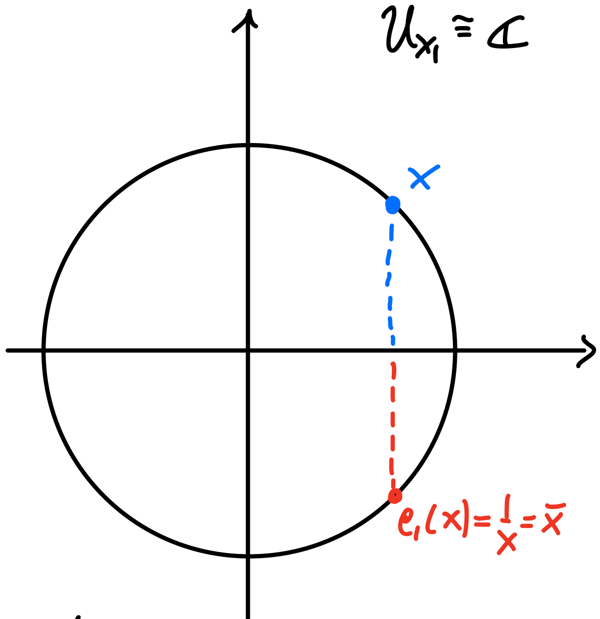
(τ: transp., σ: 3-cycle)

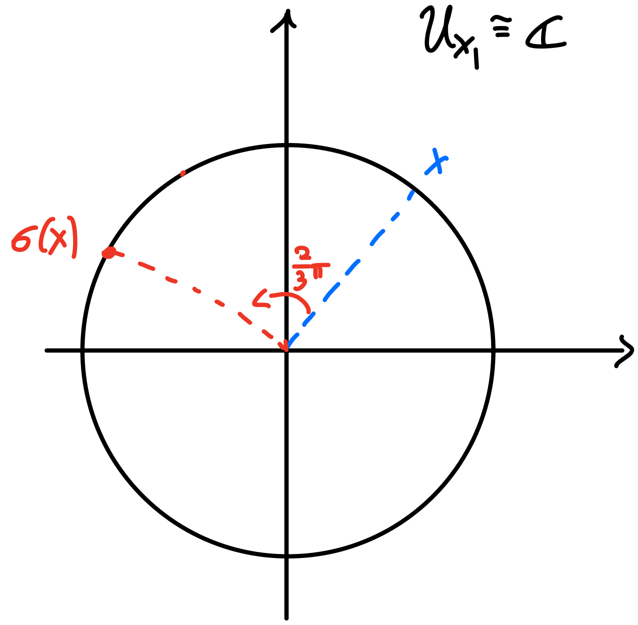
$$\tau : X \longrightarrow X, \quad [x_0, x_1] \longmapsto [x_1, x_0] \qquad , \qquad \sigma : X \longrightarrow X, \quad [x_0, x_1] \longmapsto [\xi_3 x_0, x_1]$$

$\xi_3 := e^{\frac{2\pi i}{3}}$ third root of unity

Locally around $U_{x_0} = \{x_0 \neq 0\}$ the action is

$$\tau : x \longmapsto \frac{1}{x} \left(= \bar{x} \text{ on } S^1\right), \qquad \sigma : x \longmapsto \xi_3^2 x$$





The points of X with a no trivial stabilizer are

$[1,1], [1,-1]$ $\quad \mathrm{Stab} = \langle \tau \rangle$

$[1,0], [0,1]$ $\quad \mathrm{Stab} = \langle \sigma \rangle$

$[1,-\xi_3^2], [1,\xi_3^2]$ $\quad \mathrm{Stab} = \langle \tau\sigma \rangle$

$[1,-\xi_3], [1,\xi_3]$ $\quad \mathrm{Stab} = \langle \tau\sigma^2 \rangle$

The action of S_3 on X define the S_3-quotient

$$\pi : X \to Y := \mathbb{P}^1(z_0, z_1)$$
$$[x_0, x_1] \mapsto [x_0^3 x_1^3, \frac{x_0^6 + x_1^6}{2}]$$

Locally on $\mathcal{U}_{z_0}$ we have

$$\underset{\frac{x_1}{x_0}}{\overset{\shortparallel}{x}} \xmapsto{\pi} \frac{x_0^6 + x_1^6}{2x_0^3 x_1^3} = \frac{1}{2}\left(x^3 + \frac{1}{x^3}\right)$$

$$d\pi_x = \frac{d}{dx}\left(\frac{1}{2}\left(x^3 + \frac{1}{x^3}\right)\right) = \frac{1}{2}\left(3x^2 - 3\frac{1}{x^4}\right) = \frac{3}{2}\frac{x^6 - 1}{x^4} = 0$$

$$\iff x^6 = 1 \iff x = 1, -1, \xi_3, \xi_3^2, -\xi_3, -\xi_3^2$$

which gives $[1,1], [1,-1]$
$[1, -\xi_3^2], [1, \xi_3^2]$
$[1, -\xi_3], [1, \xi_3]$

Instead, if we restrict on

$$\mathcal{U}_{x_0} \setminus \{x_0^6 + x_1^6 = 0\} \xrightarrow{\pi} \mathcal{U}_{z_1}$$
$$\underset{\frac{x_1}{x_0}}{\overset{\shortparallel}{x}} \longmapsto 2\frac{x_0^3 x_1^3}{x_0^6 + x_1^6} = 2 \cdot \frac{x^3}{1 + x^6}$$
$$\Rightarrow d\pi_x = 0 \iff x^2 = 0 \Rightarrow [1,0]$$

In a similar way we obtain $[0,1]$ using $\mathcal{U}_{x_1}$.

Thus, $\mathrm{Ram}(\pi) = [1,1] + [1,-1] + 2[1,0] + 2[0,1] + [1,-\xi_3^2] + [1,\xi_3^2]$
$[1,-\xi_3] + [1,\xi_3]$

$\mathrm{Branch}(\pi) = [1,1] + [1,-1] + 2[0,1]$

$$\Rightarrow R = \underbrace{[1,1] + [1,-1]}_{R_\tau} + \underbrace{[1,0] + [0,1]}_{R_\sigma} + \underbrace{[1,-\xi_3^2] + [1,\xi_3^2]}_{R_{\sigma\tau\sigma^2}} + \underbrace{[1,-\xi_3] + [1,\xi_3]}_{R_{\sigma^2\tau\sigma}}$$

$D = [1,1] + [1,-1] + 2[0,1]$

PROBLEM The fibre of $[1,1]$ consists of $[1,1], [1,\xi_3^2], [1,\xi_3]$, and they have different stabilizers, although conjugated to each other.

This gives a problem: the R_g are not in general a union of orbits, so the divisors D_g are not well defined as in the abelian case! This is one of the first difficulties to study non-abelian coverings.

Let us remind the irreducible representations of S_3: we have χ_{triv}, sgn and $\mu = \frac{1}{2}(\chi_{reg} - \mathrm{sgn} - 1)$ whose irreducible rep is

$$\rho_\mu(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_\mu(\sigma) = \begin{pmatrix} \xi_3^2 & 0 \\ 0 & \xi_3 \end{pmatrix}$$

check:

$$\tau\sigma = \sigma^2\tau \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_3^2 & 0 \\ 0 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & \xi_3 \\ \xi_3^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi_3 \\ \xi_3^2 & 0 \end{pmatrix}$$

Let us consider now the sheaf $\pi_* \mathcal{O}_X$ on Y, we want to prove that it is a locally free sheaf of rank 6 on Y.

We choose the coordinate charts U_{z_0} and U_{z_1} on Y: $\pi_* \mathcal{O}_X(U_{z_1}) = \mathcal{O}_X(\pi^{-1}(U_{z_1})) = \mathcal{O}_X(U_{x_0} \cap U_{x_1}) = \mathbb{C}[x, \frac{1}{x}]$

where $x := \dfrac{x_1}{x_0}$.

By construction, $G = S_3$ acts naturally on

$\pi_* \mathcal{O}_X(U_{Z_1})$ sending $e_1: x \longmapsto \frac{1}{x}$
$e_2: x \longmapsto \xi_3 x$

Thus, we have a representation of G on the space $\mathbb{C}[x, \frac{1}{x}]$. Let us determine its isotypic components W^η, $\eta \in \mathrm{Irr}(G)$ using Reynold Operator.

$S_3 = \{1, \tau, \tau\sigma, \tau\sigma^2, \sigma, \sigma^2\}$

$$\pi_{\chi_{triv}}(x^3) = \frac{1}{6}\left(x^3 + \frac{1}{x^3} + \frac{1}{x^3} + \frac{1}{x^3} + x^3 + x^3\right)$$

$$\Rightarrow \quad W^{\chi_{triv}} = \mathbb{C}[x^3 + \frac{1}{x^3}] \cdot 1.$$

and so on...

We notice that

$x, \frac{1}{x}$ generate an $\mathcal{O}_Y(U_0)$-invariant subsp. of $\mathbb{C}[x, \frac{1}{x}]$ with character μ.

$\frac{1}{x^2}, x^2$ generate an $\mathcal{O}_Y(U_0)$-inv. subsp. of $\mathbb{C}[x, \frac{1}{x}]$ with charct. μ.

Thus, the action of G on $\mathbb{C}[x, \frac{1}{x}]$ is the regular representation, and

$$\mathbb{C}[x, \tfrac{1}{x}] \overset{(*)}{=} \mathcal{O}_Y(U_0)\cdot \overset{s_{triv}}{\overset{\shortparallel}{1}} \oplus \mathcal{O}_Y(U_0)\overset{s_{sgn}}{\overset{\shortparallel}{\left(x^3 - \frac{1}{x^3}\right)}} \oplus \left(\mathcal{O}_Y(U_0)\cdot \overset{s_1}{\overset{\shortparallel}{x}} \oplus \mathcal{O}_Y(U_0)\cdot \overset{s_2}{\overset{\shortparallel}{\frac{1}{x}}}\right)$$
$$\oplus \left(\mathcal{O}_Y(U_0)\cdot \overset{s_3}{\overset{\shortparallel}{\frac{1}{x^2}}} \oplus \mathcal{O}_Y(U_0)\cdot \overset{s_4}{\overset{\shortparallel}{x^2}}\right)$$

Let us study $\pi_* \mathcal{O}_X(\mathcal{U}_{z_1})$:

$$\pi_* \mathcal{O}_X(\mathcal{U}_{z_1}) = \mathcal{O}_X(\underbrace{\pi^{-1}(\mathcal{U}_{z_1})}_{\{x_0^6+x_1^6\neq 0\}}) \overset{(*)}{=} \mathbb{C}\Big[\underbrace{\frac{ax_0-x_1}{ax_0+x_1}}_{t}, \underbrace{\frac{ax_0+x_1}{ax_0-x_1}}_{1/t}, \underbrace{\frac{a^3x_0-x_1}{a^3x_0+x_1}}_{\omega}, \underbrace{\frac{a^3x_0+x_1}{a^3x_0-x_1}}_{1/\omega}, \underbrace{\frac{a^5x_0-x_1}{a^5x_0+x_1}}_{y}, \underbrace{\frac{a^5x_0+x_1}{a^5x_0-x_1}}_{1/y}\Big] = \mathbb{C}[t, 1/t, \omega, \tfrac{1}{\omega}, y, \tfrac{1}{y}]$$

$t^6 = -1 \Rightarrow t^{12} = 1$

$\Rightarrow$ Let a be the first 12-root of the unity.

Then $a, a^3, a^5, -a, -a^3, -a^5$ are the roots of $t^6=-1$

$\Rightarrow x_0^6 + x_1^6 = (ax_0+x_1)(ax_0-x_1)(a^3x_0+x_1)(a^3x_0-x_1)(a^5x_0+x_1)(a^5x_0-x_1)$

The action of S_3 on the variables is the following :

$$\tau \cdot t = \frac{ax_1 - x_0}{ax_1+x_0} = -\frac{x_0 - ax_1}{x_0+ax_1} = -\frac{a(\frac{1}{a}x_0 - x_1)}{a(\frac{1}{a}x_0+x_1)} = -\frac{-(a^5x_0+x_1)}{-(a^5x_0-x_1)} = -\frac{1}{y}$$

$$\tau \cdot \omega = \frac{a^3x_1 - x_0}{a^3x_1+x_0} = -\frac{x_0 - a^3x_1}{x_0+a^3x_1} = -\frac{a^3(\frac{1}{a^3}x_0 - x_1)}{a^3(\frac{1}{a^3}x_0+x_1)} = -\frac{-(a^3x_0+x_1)}{-(a^3x_0-x_1)} = -\frac{1}{\omega}$$

$$\tau \cdot y = \frac{a^5x_1 - x_0}{a^5x_1+x_0} = -\frac{x_0 - a^5x_1}{x_0+a^5x_1} = -\frac{a^5(\frac{1}{a^5}x_0 - x_1)}{a^5(\frac{1}{a^5}x_0+x_1)} = -\frac{-(ax_0+x_1)}{-(ax_0-x_1)} = -\frac{1}{t}$$

$$\sigma \cdot t = \frac{a\xi_3x_0 - x_1}{a\xi_3x_0+x_1} = \frac{a^5x_0-x_1}{a^5x_0+x_1} = y$$

$$\begin{cases} \xi_3 = a^4 \\ \xi_3^2 = a^8 \end{cases}$$

$$\sigma \cdot \omega = \frac{a^3\xi_3x_0 - x_1}{a^3\xi_3x_0+x_1} = \frac{-ax_0-x_1}{-ax_0+x_1} = \frac{ax_0+x_1}{ax_0-x_1} = \frac{1}{t}$$

$$\sigma \cdot y = \frac{a^5\xi_3x_0 - x_1}{a^5\xi_3x_0+x_1} = \frac{-a^3x_0-x_1}{-a^3x_0+x_1} = \frac{a^3x_0+x_1}{a^3x_0-x_1} = \frac{1}{\omega}$$

$\overset{f_{inv}}{\overset{\shortparallel}{1}}$ is the invariant fuction of trivial charact. 1
[another one is $t-\frac{1}{t}+y-\frac{1}{y}-(w-\frac{1}{w})$]

$\underset{f_{sgn}}{\underset{\shortparallel}{t+\frac{1}{t}+w+\frac{1}{w}+y+\frac{1}{y}}}$ is the inv. fuction of charact. sgn

We obtain this by projecting t on V_{sgn}
$t \longmapsto \sum_g \mathrm{sgn}(g)\cdot g\cdot t$

$\overset{f_1}{\overset{\shortparallel}{-\frac{1}{y}-\zeta_3^2\frac{1}{t}-\zeta_3 w}}, \overset{f_2}{\overset{\shortparallel}{t+\zeta_3^2 y+\zeta_3\frac{1}{w}}}$ generate an inv. subspace of character μ.

$\overset{f_3}{\overset{\shortparallel}{t+\zeta_3 y+\zeta_3^2\frac{1}{w}}}, \overset{f_4}{\overset{\shortparallel}{-\frac{1}{y}-\zeta_3\frac{1}{t}-\zeta_3^2 w}}$ generate an inv. subspace of character μ.

Thus, the action of G on $\mathbb{C}[t,\frac{1}{t},w,\frac{1}{w},y,\frac{1}{y}]$ regular representation, and

$$\begin{aligned}\pi_*\mathcal{O}_X(\mathcal{U}_{z_1}) = {} & \mathcal{O}_Y(\mathcal{U}_{z_1})\cdot 1 \oplus \mathcal{O}_Y(\mathcal{U}_{z_2})\cdot\left(t+\frac{1}{t}+w+\frac{1}{w}+y+\frac{1}{y}\right)\oplus \\ & \oplus\left[\mathcal{O}_Y(\mathcal{U}_{z_1})\cdot\left(-\frac{1}{y}-\zeta_3^2\frac{1}{t}-\zeta_3 w\right)\oplus\mathcal{O}_Y(\mathcal{U}_{z_1})\cdot\left(t+\zeta_3^2 y+\zeta_3\frac{1}{w}\right)\right] \\ & \oplus\left[\mathcal{O}_Y(\mathcal{U}_{z_1})\cdot\left(t+\zeta_3 y+\zeta_3^2\frac{1}{w}\right)\oplus\mathcal{O}_Y(\mathcal{U}_{z_1})\cdot\left(-\frac{1}{y}-\zeta_3\frac{1}{t}-\zeta_3^2 w\right)\right]\end{aligned}$$

Let us compute the cocycles of $\pi_*\mathcal{O}_X$ to understand which locally free sheaf is on $\mathcal{Y}=\mathbb{P}^1$:

$$\bigoplus_{i=1}^{6}\mathcal{O}_Y(\mathcal{U}_0\cap\mathcal{U}_1) \longrightarrow \pi_*\mathcal{O}_X(\mathcal{U}_0\cap\mathcal{U}_1) \longrightarrow \bigoplus_{i=1}\mathcal{O}_Y(\mathcal{U}_1\cap\mathcal{U}_0)$$

$$\begin{pmatrix}\alpha_{inv}\\ \alpha_{sgn}\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix} \longmapsto \alpha_{inv}\cdot f_{inv}+\alpha_{sgn}\cdot f_{sgn}+\alpha_1\cdot f_1+\alpha_2\cdot f_2+\alpha_3\cdot f_3+\alpha_4\cdot f_4 \longmapsto \underset{?}{\underset{\shortparallel}{g_{10}}}\cdot\begin{pmatrix}\alpha_{inv}\\ \alpha_{sgn}\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}$$

$g_{inv} = 1 = f_{inv}$.

$g_{sgn} = z^3 - \frac{1}{z^3} = \frac{g_{sgn}}{f_{sgn}} \cdot f_{sgn} = -\frac{1}{6} \frac{x_0^6 + x_1^6}{x_0^3 x_1^3} = -\frac{1}{6} \frac{z_1}{z_0}$

What about g_1, g_2, g_3, g_4 in fuction of f_1, f_2, f_3, f_4?

$\boxed{g_1 = z}$ Now we need to write $z = \frac{x_1}{x_0}$ as a combination of these 4 invariant fuctions with coefficients in $\mathcal{O}_Y(U_0 \cap U_1)$.

$$z = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 \qquad \alpha_i \in \mathcal{O}_Y(U_0 \cap U_1)$$

$$\xi_3^2 z = 6 \cdot z = \alpha_1 \xi_3^2 f_1 + \alpha_2 \xi_3 f_2 + \alpha_3 \xi_3^2 f_3 + \alpha_4 \xi_3 f_4$$

$$z = \alpha_1 f_1 + \alpha_2 \xi_3 f_2 + \alpha_3 f_3 + \alpha_4 \xi_3 f_4 \qquad \Rightarrow \alpha_2 f_2 + \alpha_4 f_4 = 0$$

$$\Rightarrow \begin{cases} z = \alpha_1 f_1 + \alpha_3 f_3 \\ \tau z = \alpha_1 \tau f_1 + \alpha_3 \tau f_3 \end{cases} \qquad \begin{pmatrix} z \\ \tau z \end{pmatrix} = \begin{pmatrix} f_1 & f_3 \\ \tau f_1 & \tau f_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} = \frac{1}{f_1 \tau f_3 - \tau f_1 \cdot f_3} \cdot \begin{pmatrix} \tau f_3 & -f_3 \\ -\tau f_1 & f_1 \end{pmatrix} \begin{pmatrix} z \\ \tau \cdot z \end{pmatrix} = \frac{1}{(\ )} \cdot \begin{pmatrix} \tau f_3 z - f_3 \tau z \\ -z \tau f_1 + f_1 \tau z \end{pmatrix}$$

$$\alpha_1 = \frac{\tau f_3 \cdot z - f_3 \tau z}{f_1 \cdot \tau f_3 - \tau f_1 \cdot f_3} = \frac{1}{12} \partial^2 \frac{x_0^6 + x_1^6}{x_0^3 x_1^3} = \frac{1}{12} \partial^2 \frac{z_1}{z_0}$$

$$\alpha_3 = \frac{-z \cdot \tau f_1 + f_1 \tau z}{f_1 \cdot \tau f_3 - \tau f_1 \cdot f_3} = \frac{1}{12} (\partial^2 - 1) \frac{x_0^6 + x_1^6}{x_0^3 x_1^3} = \frac{1}{12} (\partial^2 - 1) \cdot \frac{z_1}{z_0},$$

$\alpha_2 = 0$

$\alpha_4 = 0$.

$\boxed{g_2 = \tau \cdot z = \frac{1}{z}}$ From the previous computation we already have $g_2 = \tau \cdot z = \alpha_1 \tau f_1 + \alpha_3 \tau f_3 = \alpha_1 f_2 + \alpha_3 f_4$

$$\Rightarrow \alpha_1^{g_2} = 0, \quad \alpha_2^{g_2} = \alpha_1, \quad \alpha_3^{g_2} = 0, \quad \alpha_4^{g_2} = \alpha_3.$$

$\boxed{g_3 = \frac{1}{z^2}}$ This has the same role as z, so that

$$\alpha_1^{g_3} = \frac{\tau \cdot f_3 \cdot \frac{1}{z^2} - f_3 \cdot \tau\left(\frac{1}{z^2}\right)}{f_1 \cdot \tau f_3 - \tau f_1 \cdot f_3} = \frac{1}{12}(a - a^3)\,\frac{x_0^6 + x_1^6}{x_0^3 x_1^3} = \frac{1}{12}(a-a^3)\frac{z_1}{z_0}$$

$$\alpha_3^{g_3} = \frac{-\frac{1}{z^2} \cdot \tau f_1 + f_1 \tau \cdot \left(\frac{1}{z^2}\right)}{f_1 \cdot \tau f_3 - \tau f_1 \cdot f_3} = -\frac{1}{12} a \,\frac{x_0^6 + x_1^6}{x_0^3 x_1^3} = -\frac{1}{12} a \cdot \frac{z_1}{z_0}$$

$$\alpha_2^{g_3} = \alpha_4^{g_3} = 0$$

$\boxed{g_4 = z^2}$

$$g_4 = \tau g_3 = \tau \frac{1}{z^2} = \alpha_1^{g_3} \tau f_1 + \alpha_3^{g_3} \tau f_3 = \alpha_1^{g_3} f_2 + \alpha_3^{g_3} f_4$$

$$\Rightarrow \alpha_1^{g_4} = 0, \quad \alpha_2^{g_4} = \alpha_1^{g_3}, \quad \alpha_3^{g_4} = 0, \quad \alpha_4^{g_4} = \alpha_3^{g_3} \quad \checkmark$$

So

$$\rho_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{6}\frac{z_1}{z_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} a^2 \frac{z_1}{z_0} & 0 & \frac{1}{12}(a-a^3)\frac{z_1}{z_0} & 0 \\ 0 & 0 & 0 & \frac{1}{12} a^2 \frac{z_1}{z_0} & 0 & \frac{1}{12}(a-a^3)\frac{z_1}{z_0} \\ 0 & 0 & \frac{1}{12}(a^2-1)\frac{z_1}{z_0} & 0 & -\frac{1}{12} a \frac{z_1}{z_0} & 0 \\ 0 & 0 & 0 & \frac{1}{12}(a^2-1)\frac{z_1}{z_0} & 0 & -\frac{1}{12} a \frac{z_1}{z_0} \end{pmatrix}$$

Conclusions

$\pi_* \mathcal{O}_X$ still decomposes as the regular representation:

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in \mathrm{Irr}(G)} (\pi_* \mathcal{O}_X)^\chi$$

where the isotypic components $(\pi_* \mathcal{O}_X)^\chi$ are of degree $\chi(1_G)^2$, namely the irreducible repr. of χ appears on $(\pi_* \mathcal{O}_X)^\chi$ exactly $\chi(1_G)$-times.

We obtain that $(\pi_* \mathcal{O}_X)^\chi$ are locally-free sheaves of degree $\chi^2(1_G)$; however, in general it is not true that they decomposes as the direct sum of $\chi(1_G)$ - locally free sheaves of deg $\chi(1_G)$. Indeed $(\pi_* \mathcal{O}_X)^\chi$ could be indecomposable. Another difference with respect to the abelian case is that $\{(\pi_* \mathcal{O}_X)^\chi\}_{\chi \in Irr(G)}$ have not anymore an operation involving them as for invertible sheaves. This makes very difficult to understand what are the relationships among $\{(\pi_* \mathcal{O}_X)^\chi\}_\chi$ and the divisors $\{R_g\}_{g \in G}$.

For these reasons, it is known a solid theory only for abelian coverings (although something for Dihedral covering has been achieved by Catanese - Perroni, 2016).